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Some remarks on Hrushovski constructions in irrational case

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Abstract

In rational case, the index of a free amalgamation class $\mathscr{C}_f$ is infinite if and only if the theory of the Fraïssé limit $\mathcal{M}_f$ of $\mathscr{C}_f$ is model complete. On the other hand, it is unknown whether the same assertion holds or not in irrational case. However, we can construct a free amalgamation class with the model complete theory from a given class. In this paper, we will introduce the constructions of free amalgamation classes with generic structures having model complete theories.

1 Introduction

Definition 1.1. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and $A \subseteq B$ be finite graphs.

(1) $\delta_\alpha(A) := |A| - \alpha|E(A)|$.

(2) $\delta_\alpha(B/A) := \delta_\alpha(B) - \delta_\alpha(A) = |B \setminus A| - |E(B) \setminus E(A)|$.

(3) $A \leq_\alpha B :\iff \delta_\alpha(X/A) > 0$ for all $A \subsetneq X \subseteq B$.

(4) $\mathscr{C}_\alpha := \{A \mid \emptyset \leq_\alpha A\}$.

We say A is *closed* in B when $A \leq_\alpha B$.

Definition 1.2. Let $A \subseteq B, C$ with $A = B \cap C$ and $D \supseteq B, C$ with $D = BC$. $D = B \otimes_A C$ if D has no edges between $B \setminus A$ and $C \setminus A$, called a *free amalgam* of B and C over A.

Definition 1.3. Suppose $\mathscr{C} \subseteq \mathscr{C}_\alpha$ is closed under ismorphism.

(1) (Hereditary Property) For all $A \in \mathscr{C}$, every $B \subseteq A$ is in $\mathscr{C}$.

(2) (Free Amalgamation Property) For all $A, B, C \in \mathscr{C}$ with $A \leq_\alpha B, C$, we have $B \otimes_A C \in \mathscr{C}$.

$\mathscr{C}$ is *free amalgamation class* if $\mathscr{C}$ has the hereditary property and the free amalgamation property.

Note that $\mathscr{C}_\alpha$ is the free amalgamation class.

Fact 1.4. If $\mathscr{C}$ is the free amalgamation class, then there is a countable graph $\mathcal{M}$ called a generic structure which has the following conditions:

(1) Every $A \subseteq_{\text{fin}} \mathcal{M}$ is in $\mathscr{C}$.

(2) For all $A \subseteq_{\text{fin}} \mathcal{M}$, there is $B \subseteq_{\text{fin}} \mathcal{M}$ such that $A \subseteq B \leq_\alpha \mathcal{M}$.

(3) For all $A, B \in \mathscr{C}$ with $A \leq_\alpha \mathcal{M}$ and $A \leq_\alpha B$, we can closedly embed B into $\mathcal{M}$ fixing A pointwise.

2 Model completeness in rational cases

Let $m, d \in \mathbb{N} \setminus \{0\}$ be relatively prime with each other and $m < d$, and $\alpha = m/d$.

Definition 2.1. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *good* if the following conditions hold:

(1) $f(0) = 0$, $f(1) \leq 1$.

(2) f is a concave, unbounded and (piecewise) smooth function.

(3) $f'(x) \leq 1/dx$.

Example 2.2. Let $\alpha = 1/2$. Then $f(x) = \log(x + 1)/2$ is good.

Fact 2.3. Let f be good and $\mathscr{C}_f := \{A \in \mathscr{C}_\alpha \mid f(|X|) \leq \delta_\alpha(X) \text{ for all } X \subseteq A\}$. Then $\mathscr{C}_f$ is the free amalgamation class.

Definition 2.4. $A \in \mathscr{C}$ is *absolutely closed* in $\mathscr{C}$ if for all $B \in \mathscr{C}$, $A \subseteq B$ implies $A \leq_\alpha B$.

Fact 2.5. Suppose that for all $A \in \mathscr{C}$, there is $C \in \mathscr{C}$ with $A \leq_\alpha C$ such that C is absolutely closed. Then the theory of the generic structure of $\mathscr{C}$ is model complete.

Fact 2.6. Let $\alpha \in \mathbb{Q}$ with $0 < \alpha < 1$ and f be Hrushovski's good concave function. Then $\mathscr{C}_f$ has the above condition, so $\mathrm{Th}(\mathcal{M}_f)$ is model complete.

3 Main Theorem

When α is irrational, it is impossible to consider good functions. We replace a condition in the definition of good functions.

Definition 3.1. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *good* if the following conditions hold:

(1) $f(0) = 0$, $f(1) \leq 1$.

(2) f is a concave, unbounded and (piecewise) smooth function.

(3) $f'(x) \leq d_\alpha^+(x)/dx$.

Where $d_\alpha^+(x) := \min\{p - q\alpha \mid p \leq x, p/q > \alpha\}$.

Fact 3.2. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and f be good. Then $\mathscr{C}_f$ is the free amalgamation class.

Fact 3.3. Each irrational α has infinitely many good functions.

To construct a free amalgamation class having good graphs, we will consider a weak notion of free amalgamation classes.

Definition 3.4. Let $n < \omega$. Suppose $\mathscr{C}_0 \subseteq \mathscr{C}_\alpha$ is closed under isomorphism and has the hereditary property. $\mathscr{C}_0$ is the *n-free amalgamation class* if for all $A, B, C \in \mathscr{C}_0$ with $A \leq_\alpha B, C$ and $D = B \otimes_A C$, we have $D \in \mathscr{C}_0$ only when $|B|, |C| < n$.

Definition 3.5. Suppose $\mathscr{C}_0 \subseteq \mathscr{C}_\alpha$ is 0-free amalgamation class. $F_\alpha(\mathscr{C}_0) := \{B_1 \otimes_A B_2 \mid A \cong A', B_i \cong B_i' \text{ and } A = B_1 \cap B_2 \leq_\alpha B_i \text{ for some } A', B_i' \in \mathscr{C}_0\}$.

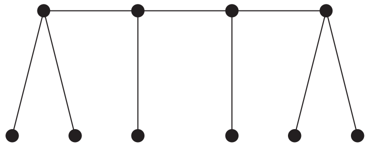
Lemma 3.6. Let $n < \omega$ and $\mathscr{C}_0$ be an n-free amalgamation class. Then $F_\alpha(\mathscr{C}_0)$ is an $n+1$-free amalgamation class. In particular, $A \in \mathscr{C}_0 \iff A \in F_\alpha(\mathscr{C}_0)$ whenever $|A| < n$.

Lemma 3.7. Let $\mathscr{C}_0$ is 0-free amalgamation class. Then $\mathscr{C} := \lim_{n\to\infty} F_\alpha^n(\mathscr{C}_0)$ is the smallest free amalgamation class containing $\mathscr{C}_0$.

Definition 3.8. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. $A \subseteq B$ is an *zero-extension* if $\delta_\alpha(B/A) = 0$ and $\delta_\alpha(X/A) > 0$ for every intermediate X between A and B.

The zero-extension is the important notion for model completeness in rational cases, but irrational α has no zero-extensions. Then we introduce another notion. $A \subseteq B$ is an *intrinsic extension* if $\delta_\alpha(B/A) \leq 0$ and $\delta_\alpha(X/A) > 0$ for every intermediate X between A and B.

Example 3.9. Let $\alpha = 1/\sqrt{5}$. Then the below relative graph is intrinsic extension because $5/9$ is the best approximation to α from below with denominator less than 10.



Lemma 3.10. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and f be good. Assume that $A \in \mathscr{C}_f$ has sufficiently many vertices. Then we can extend A to C which is not necessarily in $\mathscr{C}_f$ but whose every proper subset is in $\mathscr{C}_f$ by using an intrinsic extension. In particular, C is absolutely closed in a smallest free amalgamation class containing $\mathscr{C}_f \cup \{C\}$.

Theorem 3.11. Suppose α is irrational and has a good function. Then there is a free amalgamation class $\mathscr{C} \subseteq \mathscr{C}_\alpha$ such that its generic structure $\mathcal{M}$ has a model complete theory.

Proof. It is enough to find an increasing sequence of unbounded free amalgamation classes $\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \cdots \subseteq \mathscr{C}_n \subseteq \cdots$ such that:

(1) For all A with at most n vertices, $A \in \mathscr{C}_n \iff A \in \mathscr{C}_{n+1}$.

(2) For each $n < \omega$, there is $C \in \mathscr{C}_{n+1}$ such that every $A \in \mathscr{C}_n$ with n vertices can be closedly embedded into C.

(3) Above C is absolutely closed in $\mathscr{C}_m$ for all $m > n$.

Let $\mathscr{C} = \bigcup_{n<\omega} \mathscr{C}_n$. By the conditions, C is absolutely closed in $\mathscr{C}$. Hence $\mathscr{C}$ has a generic structure $\mathcal{M}$ having a model complete theory by Lemma ???. □

4 Future works

Conjecture 4.1. There is an irrational α such that $\mathrm{Th}(\mathcal{M}_f)$ is model complete for all good f.

Problem 4.2. For all α having an unbounded free amalgamation class but no good function, is there $\mathscr{C}$ such that its theory of the generic structure is model complete?

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